More structures on the space of modular forms;

Petersson inner product

and

Hecke operators

Petersson inner product.

Definition: Let $M_k(\Gamma)$ be the space of holomorphic modular forms of weight k and congruence subgroup Γ.
Let $S_k(\Gamma)$ be the analogous space of cusp forms.

The map $\langle \cdot , \cdot \rangle : \mathcal{M}_k(\Gamma) \times S_k(\Gamma) \to \mathbb{C}$

$$\langle f, g \rangle := \int_{\mathcal{F}_\Gamma} f(z)\,\overline{g(z)}\,(\operatorname{Im} z)^k\, d\mu(z)$$

is called **Petersson inner product**, where
$\mathcal{F}_\Gamma$ is a fundamental domain of $\Gamma \backslash \mathfrak{h}$

$d\mu(z) = y^{-2}\,dx\,dy \qquad z = x + iy, \quad x, y \in \mathbb{R}.$

$SL_2(\mathbb{R})$ invariant measure

Questions:

① Does this integral converge?

② Does the definition depend on our choice of $\mathcal{F}_\Gamma$?

Lemma: The integral is absolutely convergent, does not depend on a choise of $\mathcal{F}_\Gamma$, and the Petersson inner product is a positive definite Hermitian form on $S_k(\Gamma)$.

Lemma: The Eisenstein series $E_k \in M_k(\Gamma_1)$ is orthogonal to $S_k(\Gamma_1)$ w.r.t. Petersson inner product.

Proof: $\Gamma_\infty = \mathrm{Stab}_\infty(\Gamma_1) = \left\{ \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \;\middle|\; n \in \mathbb{Z} \right\}.$

$$1\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (c\tau + d)^{-k}.$$

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} 1\Big|_k \gamma \overset{\text{why?}}{=} \sum_{(c,d)=1} \frac{1}{(c\tau+d)^k} = \zeta^{-1}(k) \sum_{(c,d) \in \mathbb{Z}^2 \setminus 0} \frac{1}{(c\tau+d)^k} \doteq E_k(\tau).$$

$$\int_{\Gamma_1 \backslash \mathfrak{h}} E_k(\tau)\, \overline{f(\tau)}\, \mathrm{Im}(\tau)^k \, d\mu(\tau) =$$

$$\int_{\Gamma_1 \backslash \mathfrak{h}} E_k(\tau)\,\overline{f(\tau)}\,\mathrm{Im}(\tau)^k\,d\mu(\tau) =$$

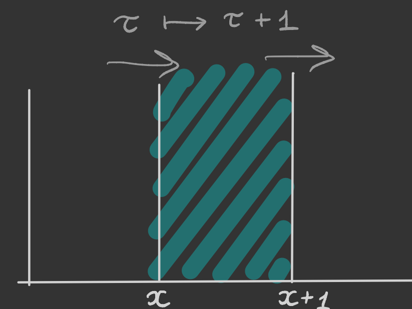
$$= \int_{\Gamma_1 \backslash \mathfrak{h}} \Big(\sum_{\Gamma_\infty \backslash \Gamma_1} 1\big|_k \gamma \Big)\,\overline{f(\tau)}\,\mathrm{Im}(\tau)^k\,d\mu(\tau)$$

$$= \int_{\Gamma_1 \backslash \mathfrak{h}} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_1} (c\tau+d)^{-k}\,\overline{(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)}\,\mathrm{Im}(\tau)^k\,d\mu(\tau)$$

$$= \int_{\Gamma_1 \backslash \mathfrak{h}} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_1} (c\tau+d)^{-k} \overline{(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)} \operatorname{Im}(\tau)^k \, d\mu(\tau)$$

$$= \int_{\Gamma_1 \backslash \mathfrak{h}} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_1} \overline{f\left(\frac{a\tau+b}{c\tau+d}\right)} \cdot \operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)^k d\mu\left(\frac{a\tau+b}{c\tau+d}\right)$$

$$= \int_{\Gamma_\infty \backslash \mathfrak{h}} \overline{f(\tau)} \cdot \operatorname{Im}(\tau)^k \, d\mu(\tau)$$



$$= \int \overline{\left(\sum_{n=1}^{\infty} c_f(n) e^{2\pi i n \tau}\right)} \operatorname{Im}(\tau)^k \, d\mu(\tau) = \int_0^\infty \int_{x_0}^{x_0+1} \left(\sum_{n=1}^{\infty} c_f(n) e^{2\pi i (x+iy) n}\right) y^{k-2} \, dx \, dy$$

$$= 0. \quad \blacksquare$$

Remark: For Γ_1 : $\dim(M_k(\Gamma_1)) - \dim(S_k(\Gamma_1)) = 1$, $k \geq 4$

$$M_k(\Gamma_1) = \mathbb{C} E_k \oplus S_k(\Gamma_1)$$

For other groups Γ be define the Eisenstein space $\mathcal{E}_k(\Gamma)$ as $S_k(\Gamma)^{\perp}$ w.r.t. Petersson product.

Poincare series

Lemma: For $n \in \mathbb{Z}_{\geq 1}$ there exists a unique cusp form $P_n \in S_k(\Gamma_1)$ such that

$$\langle f, P_n \rangle_{\text{Petersson}} = c_f(n) \text{ for all } f \in S_k(\Gamma_1).$$

Remark: ① P_n can be computed explicitly for $k \geq 4$ $P_n(z) = c_{n,k} \cdot \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} e^{2\pi i n z}\big|_k \gamma$

② From this representation of P_n the Fourier coefficients of P_n can be obtained

③ This Fourier coefficients are represented as a sum of certain infinite series, and are transcendental in general

Hecke operators.

Let us do the case of Γ_1 first.

2 definitions:

▶ View modular forms as a function of lattices

Let F be a function on the set of lattices $\Lambda \subset \mathbb{C}$.

Suppose that F satisfies:

$$F(\alpha \cdot \Lambda) = \alpha^{-k} F(\Lambda), \quad \alpha \in \mathbb{C}^\times$$

Define $f : \mathfrak{h} \to \mathbb{C}$ by $f(\tau) := F(\mathbb{Z} + \tau\mathbb{Z})$.

Let $\tau \in \mathfrak{h}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = F\left(\mathbb{Z} + \frac{a\tau+b}{c\tau+d}\mathbb{Z}\right) = (c\tau+d)^k \cdot F\Big((c\tau+d)\mathbb{Z} + (a\tau+b)\mathbb{Z}\Big)$$

$$= (c\tau+d)^k F(\mathbb{Z} + \tau\mathbb{Z}) = (c\tau+d)^k f(\tau)$$

For $m \in \mathbb{Z}_{\geq 1}$ we define the m-th Hecke operator

$$T_m F(\Lambda) := \sum_{\substack{\Lambda' \subset \Lambda \\ [\Lambda : \Lambda'] = m}} F(\Lambda')$$

Now we need to translate this to the language of f.

Suppose $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\Lambda' \subset \Lambda$ $[\Lambda : \Lambda'] = m$, then

$\Lambda' = (a\tau + b)\mathbb{Z} + (c\tau + d)\mathbb{Z}$ for some

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}) \quad \text{with} \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = m.}$$

$$T_m f(\tau) = \underbrace{m^{k-1}}_{\text{normalization}} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \backslash \mathcal{M}_m} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right)$$

set of matrices

▷ This definition is equivalent to:

$$(1) \quad T_m f = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} \frac{1}{d^k} \sum_{b \bmod d} f\left(\frac{az+b}{d}\right)$$

Lemma: The collection $\Delta_m := \{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad=m,\ 0 \leq b < d \}$ is a complete set of right coset representatives of $\Gamma_1 \setminus \mathcal{M}_m$. $\mathcal{M}_m = \dot{\bigcup}_{M \in \Delta_m} \Gamma_1 \cdot M$

On the level of Fourier coefficients:

Suppose $f(\tau) = \sum_{n=0}^{\infty} c_f(n) e^{2\pi i n \tau}$. Then

$$T_m f(\tau) = \sum_{n \geq 0} \Big(\sum_{\substack{r \mid (m,n) \\ r>0}} r^{k-1} c_f(mn/r^2) \Big) e^{2\pi i n \tau}$$

Lemma: a) If $(m_1, m_2) = 1$ ($m_1, m_2 \in \mathbb{Z}_{\geq 1}$) then $T_{m_1} \cdot T_{m_2} = T_{m_1 \cdot m_2}$

b) $T_m \cdot T_n = T_n \cdot T_m \qquad m, n \in \mathbb{Z}_{\geq 1}$

Proof: Exercise.

Reference: H. Iwaniec

„Topics in Classical Automorphic Forms"

▷▷ Another definition: Double coset operator

Read „ A first course in modular forms"

Section 5.1.

Hecke operators are defined for $\Gamma_1(N)$.

Today: $\Gamma = \Gamma_1$

Theorem: The Hecke operator T_m is self-adjoint with respect to Petersson inner product.

Proof:

„Topics in Classical Automorphic Forms"

<u>Step 1</u>: Action of Hecke operators on Poincare series

$$(T_n P_m)(z) = \sum_{d|(m,n)} (n/d)^{k-1} P_{mnd^{-2}}$$

<u>Step 2</u>: $m, n \geq 1$

$$m^{k-1} T_n P_m = n^{k-1} T_m P_n$$

<u>Step 3</u>: $\langle T_n P_m, P_\ell \rangle = \langle P_m, T_n P_\ell \rangle$ $m \in \mathbb{Z}_{\geq 0}, n, \ell \in \mathbb{Z}_{\geq 1}$

Lemma: For $k > 2$, $m \geq 0$ and $n \geq 1$ we have

$$T_n P_m(z) = \sum_{d \mid (m,n)} (n/d)^{k-1} P_{mnd^{-2}}.$$

Proof:

We notice that

$$(T_n P_m)(z) = \sum_{g \in \Gamma_\infty \backslash \mathcal{M}_n} (c_g z + d_g)^{-k} e^{2\pi i m\, g(z)}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be any sets of right coset representatives of $\Gamma_\infty \backslash \Gamma$ and $\Gamma \backslash \mathcal{M}_n$ respectively.

Then naturally $\mathcal{B}\mathcal{A}$ is a set of right coset representatives of $\Gamma_\infty \backslash \mathcal{M}_n$.

$$T_n P_m(z) = n^{k-1} \sum_{\beta \in \mathcal{B}} \sum_{\alpha \in \mathcal{A}} (c_{\beta\alpha} z + d_{\beta\alpha})^{-k} e^{2\pi i m \beta\alpha(z)}$$

$$= n^{k-1} \sum_{ad=n} d^{-k} \sum_{b \,(\mathrm{mod}\, d)} \sum_{\alpha \in \mathcal{A}} (c_\alpha z + d_\alpha)^{-k} e^{2\pi i m \frac{a\alpha(z)+b}{m}}$$

$$= n^{k-1} \sum_{\substack{ad=n \\ d \mid m}} d^{1-k} \sum_{\alpha \in \mathcal{A}} (c_\alpha z + d_\alpha)^{-k} e^{2\pi i \frac{am}{d} \alpha(z)}$$

$$= n^{k-1} \sum_{\substack{ad=n \\ d \mid m}} d^{1-k} P_{\frac{am}{d}}(z). \quad \blacksquare$$

Corollary: $m^{k-1}\, T_n P_m = n^{k-1}\, T_m P_n$

Corollary: For $m, n \geq 1$ and $f \in M_k(\Gamma_1)$

$$m^{k-1} \langle T_n f, P_m \rangle = n^{k-1} \langle T_m f, P_n \rangle$$

Theorem: For all $f \in M_k(\Gamma_1)$, $g \in S_k(\Gamma_1)$

$$\langle T_n f, g \rangle = \langle f, T_n g \rangle$$

Proof: It suffices to prove this formula for

$f = P_m$, $g = P_\ell$ $\quad m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$

Assume $m \neq 0$

$$\langle T_n P_m, P_\ell \rangle = \langle T_n P_\ell, P_m \rangle = \langle P_m, T_n P_\ell \rangle \quad \blacksquare$$

Definition: A cusp form $f \in S_k(\Gamma_1)$ is a **Hecke eigenform** if it is an eigenvector for all Hecke operators T_m, $m \in \mathbb{Z}_{\geq 1}$.

Theorem: The space $S_k(\Gamma_1)$ has basis consisting of Hecke eigenforms.

Lemma: Suppose that $T_m f = \lambda_m f$, $m \in \mathbb{Z}_{\geq 1}$. Then $c_f(m) = \lambda_m c_f(1)$.

Theorem (Multiplicity one) Suppose that $f, g \in S_k(\Gamma_1)$ are Hecke eigenforms and $T_m f = \lambda_m f$, $T_m g = \lambda_m g$ for all $m \in \mathbb{Z}_{\geq 1}$. Then $f = c \cdot g$ for some $c \in \mathbb{C}^\times$.

Eigenfunctions of Hecke operators.

Examples:

1. Eisenstein series ← This can be checked from Fourier expansion
2. Ramanujan Δ function

(1) $\mathbb{C} E_k = S_k^{\perp} \Rightarrow T_m$-s are self-adjoint

$T_m(S_k) = S_k$

Therefore $T_m(\mathbb{C} E_k) \subseteq \mathbb{C} \cdot E_k$

(2) $\dim(S_{12}) = 1 \Rightarrow T_m(\mathbb{C} \cdot \Delta) \subseteq \mathbb{C} \cdot \Delta$

$\exists ?\, m$ such that $T_m \Delta = 0$

Let $f \in M_k(\Gamma_1)$ be a Hecke eigenform.
We normalize f s.t $c_f(1) = 1$.
Then

$$c_f(m \cdot n) = c_f(m) \cdot c_f(n) \quad \text{if } (m,n) = 1, \quad m,n \in \mathbb{Z}_{\geq 1}$$

$$c_f(p^{\nu+1}) = c_f(p)\, c_f(p^{\nu}) - p^{k-1} c_f(p^{\nu-1}) \quad \text{if } p \text{ prime}, \nu \geq 1$$

Definition: The Hecke L-series of $f \in S_k(\Gamma_1)$ is

$$L(f,s) := \sum_{n=1}^{\infty} \frac{c_f(n)}{n^s}$$

Theorem (Euler product) If f is a Hecke eigenform, then

$$L(f,s) = \prod_{p \text{ prime}} \left(1 + \frac{c_f(p)}{p^s} + \frac{c_f(p^2)}{p^{2s}} + \dots \right) = \prod_{p \text{ prime}} \left(1 - c_f(p) p^{-s} + p^{k-1-2s}\right)^{-1}$$

Lemma: Suppose that f is a cusp form of weight $k \in 2\mathbb{Z}_{>0}$ and group Γ_1, and f has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} c_f(n) e^{2\pi i n z}$$

Then $c_f(n) = \underline{O}(n^{k/2})$ as $n \to \infty$

Proof: Let $f \in S_k(\Gamma_1)$.

Then the function $|f(z)|\, \mathrm{Im}(z)^{k/2} =: g(z)$ is Γ_1-invariant.

Moreover, $g(z)$ is bounded as $z \in \mathfrak{h}$

Suppose $|g(z)| < C$, $z \in \mathfrak{h}$ for some $C \in \mathbb{R}_{>0}$

We estimate the Fourier coefficients of f in the following way:

$y_0 > 0$

$$C_f(n) = \int_{iy_0}^{iy_0+1} f(z)\, e^{-2\pi i n z}\, dz$$

$$|C_f(n)| = \left| \int_{iy_0}^{iy_0+1} f(z)\, e^{-2\pi i n z}\, dz \right| = \left\{ \begin{array}{l} \text{we choose} \\ y_0 = \frac{1}{n} \end{array} \right\} = \left| \int_{i/n}^{i/n+1} f(z)\, \mathrm{Im}(z)^{k/2} \cdot \mathrm{Im}(z)^{-k/2}\, e^{-2\pi i n z}\, dz \right|$$

$$\leq \int_{i/n}^{i/n+1} \left| f(z)\, \mathrm{Im}(z)^{k/2} \right| \cdot n^{k/2} \cdot e^{2\pi} \cdot \left| e^{-2\pi i n \cdot \mathrm{Re}(z)} \right| dz \leq C \cdot e^{2\pi} \cdot n^{k/2}$$

This finishes the proof of the lemma $\blacksquare$

Estimate $c_f(n) = O(n^{k/2})$ implies:

$\mathcal{L}(f,s)$ converges absolutely in the half-plane

$$\mathrm{Re}(s) > 1 + k/2$$

Definition: $\mathcal{L}^*(f,s) := (2\pi)^{-s}\,\Gamma(s)\,L(f,s)$

Theorem: Let f be a cusp form of weight k on the full modular group. Then the L-series $L(f,s)$ extends to an entire function of s and satisfies functional equation:

$$L^*(f,k-s) = (-1)^{k/2} L^*(f,s)$$

Proof: $L^*(f,s) = (2\pi)^{-s}\,\Gamma(s)\,L(f,s) =$

We use the identity $\displaystyle\int_0^\infty t^{s-1} e^{-2\pi n t}\,dt = (2\pi)^{-s}\,\Gamma(s)\,n^{-s}$

$$= \sum_{n=1}^{\infty} c_f(n) \int_0^\infty t^{s-1} e^{-2\pi n t}\,dt =$$

$$= \int_0^\infty t^{s-1} f(it)\,dt \quad , \qquad \mathrm{Re}(s) > \frac{k}{2}+1$$

This integral converges absolutely for all $s \in \mathbb{C}$, because f is a cusp form and

$|f(it)| = O(e^{-2\pi t}),\ t\to\infty$ and $|f(it)| = O(t^{-k} f^{-\frac{2\pi}{t}}),\ t\to 0$

Now we use the modularity of f

$$\int_0^\infty t^{s-1} f(it)\,dt = \int_0^\infty t^{s-1}\, t^{-k} f\left(\frac{i}{t}\right) dt$$

change of variables

$t \mapsto \frac{1}{u}$

$$= \int_0^\infty u^{1-s}\, u^{k}\, f(iu)\, u^{-2}\, du$$

$$= \int_0^\infty u^{k-s-1} f(iu)\,du = L^*(f, k-s) \quad \blacksquare$$

Ramanujan conjecture

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z}$$

In 1916 S. Ramanujan conjectured that

$$|\tau(p)| \leq 2p^{11/2} \text{ for all primes } p.$$

$$|\tau(n)| \leq d(n)\, n^{11/2},$$

where $d(n)$ is the number of positive divisors of n

Ramanujan-Petersson conjecture

for holomorphic cusp forms

$$|\lambda(n)| \leq d(n)$$

Here $\lambda(n)$ is a properly normalised Hecke eigenvalue

Theorem (Deligne 1971), (Deligne-Serre 1974)

↑ for weight 1